What is the meaning and purpose of *e* and the natural logarithm?

Part A: Finding the rate of change (derivative) of an exponential function

$\frac{d}{dx}(a^x) = ? \quad , \quad \forall \; a > 0$

$$\frac{d}{dx}(a^x) = \lim_{\Delta x \to 0}\left[\frac{a^{x+\Delta x} - a^x}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{a^x \cdot a^{\Delta x} - a^x}{\Delta x}\right] = a^x \lim_{\Delta x \to 0}\left[\frac{a^{\Delta x} - 1}{\Delta x}\right]$$

$$\therefore \frac{d}{dx}(a^x) = \underbrace{a^x}_{\text{function of } x \text{ only}} \times \underbrace{\lim_{\Delta x \to 0}\left[\frac{a^{\Delta x} - 1}{\Delta x}\right]}_{\text{function of } \Delta x \text{ only}} \qquad (1)$$

Numerically find the value for a for which $\lim_{\Delta x \to 0}\left[\frac{a^{\Delta x}-1}{\Delta x}\right] = 1$, and call (**define**) that value $\boldsymbol{e}$.

So: $\lim_{\Delta x \to 0}\left[\frac{e^{\Delta x}-1}{\Delta x}\right] = 1$

$$\therefore \frac{d}{dx}(e^x) = e^x \times \lim_{\Delta x \to 0}\left[\frac{e^{\Delta x} - 1}{\Delta x}\right] = e^x \times 1 = e^x$$

To find $\frac{d}{dx}(a^x)$, first write a in terms of e:

$a = e^{\text{the exponent to which } e \text{ should be raised to give } a}$ (This is the definition of a logarithm.)

$\therefore a = e^{\log_e a}$

Define the abbreviation: $\mathbf{ln}\, y = \log_e y$.

$\therefore a = e^{\ln a}$

$$\therefore a^{\Delta x} = \left(e^{\ln a}\right)^{\Delta x} = e^{\Delta x \cdot \ln a} = e^{\ln a \cdot \Delta x} \qquad (2)$$

From (1): $\frac{d}{dx}(a^x) = a^x \lim_{\Delta x \to 0}\left[\frac{a^{\Delta x}-1}{\Delta x}\right]$

Substitute for $a^{\Delta x}$ by using (2):

$$\therefore \frac{d}{dx}(a^x) = a^x \times \lim_{\Delta x \to 0}\left[\frac{e^{\ln a \cdot \Delta x} - 1}{\Delta x}\right]$$

Differentiate the numerator as well the denominator of the limit with respect to Δx.
According to *L'Hôpital's Rule*, the value of the limit will remain the same:

$$\therefore \frac{d}{dx}(a^x) = a^x \times \lim_{\Delta x \to 0}\left[\frac{\frac{d}{dx}\left(e^{\ln a \cdot \Delta x}\right) - 0}{1}\right] \qquad (3)$$

To find $\frac{d}{dx}\left(e^{\ln a \cdot \Delta x}\right)$, use the chain rule (for a function of a function):

$$\therefore \frac{d}{dx}\left(e^{\ln a \cdot \Delta x}\right) = e^{\ln a \cdot \Delta x} \times \ln a = a^{\Delta x} \times \ln a \qquad (4)$$

Substitute (4) into (3), and note that $a^{\Delta x} \to 1 \; as \; x \to 0$:

$$\therefore \frac{d}{dx}(a^x) = a^x \times \lim_{\Delta x \to 0}\left[\frac{a^{\Delta x} \times \ln a - 0}{1}\right] = a^x \times \ln a \quad , \quad \forall \; a > 0 \qquad (5)$$

Part B: Finding the accumulation (integral) of an hyperbolic function

$\int \frac{1}{x} dx = ?$ **NB:** This question can *not* be asked for a range a x that contains $x = 0$.
So the following only applies to $x > 0$ or $x < 0$.

Let: $y = \int \frac{1}{x} dx$

So: $\frac{dy}{dx} = \frac{1}{x}$

Let us test whether $\ln|x|$ will give the required derivative of $\frac{1}{x}$ (we are integrating 'by inspection'):

$$\frac{d}{dx}(\ln|x|) = \lim_{\Delta x \to 0} \left[\frac{\ln|x + \Delta x| - \ln|x|}{\Delta x} \right]$$

We cannot write $\ln|x + \Delta x|$ in a different way, so the above plan is not helping us.
Let us rather try making a substitution:

(i) Define for $x > 0$: $y = \ln|x| = \ln(x) \quad , \quad \forall \ x > 0$

$\therefore e^y = x$ (6a)

Do 'implicit differentiation' of (6a):

$$\therefore \frac{d}{dx}(e^y) = \frac{d}{dx}(x) = 1$$

Find the derivative of e^y with respect to x by using the chain rule (for a function of a function):

$$\frac{d}{dx}(e^y) = e^y \frac{dy}{dx} = 1$$

Re-arrange and substitute for e^y using (6a):

$$\therefore \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad , \quad \forall x > 0$ (7a)

(ii) Define for $x < 0$: $y = \ln|x| = \ln(-x) \quad , \quad \forall \ x < 0$

$\therefore e^y = -x$ (6b)

$$\therefore \frac{d}{dx}(e^y) = \frac{d}{dx}(-x) = -1$$

$$\frac{d}{dx}(e^y) = e^y \frac{dy}{dx} = -1$$

Re-arrange and substitute for e^y using (6b):

$$\therefore \frac{dy}{dx} = \frac{1}{e^y} = \frac{-1}{-x} = \frac{1}{x}$$

$\therefore \frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad , \quad \forall x < 0$ (7b)

(iii) In summary: $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad , \quad \forall \ x > 0, x < 0$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad , \quad \forall \ x \neq 0$ (8)