RME and its challenges

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Overview

Realistic Mathematics Education (RME)

Shortcomings RME innovation Netherlands

Footholds for improvement

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A 'layman's view on instruction

- How do people learn?
- Layman's view: By making connections between what is known and what has to be learned
- Thus: How do people learn mathematics?

→Learning Mathematics: making connections with an abstract, formal body of knowledge

- Problem: Gap between the knowledge of the students and the abstract, formal body of knowledge

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What makes mathematics so difficult?

- The problem is not in the abstract character of mathematics as such
- It is the gap between the abstract knowledge of the teachers and the experiential knowledge of the students
  - Teachers and textbook authors tend to (mis)take their own more abstract mathematical knowledge for an objective body of knowledge with which the students can make connections
- Students cannot make connections with knowledge that is not there for them

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The new mathematical knowledge does not exist yet

- Young children don't understand the question: "How much is 4+4?

  Even though they know that "4 apples and 4 apples makes 8 apples"
- Van Hiele Levels:
  - Ground level: Number tied to countable objects: "four apples"
  - Higher level: 4 is associated with number relations: 4 = 2+2 = 3+1 = 5-1 = 8:2
  - mathematical object

  compare Sfard (1991) structural - operational

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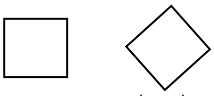
Caveat

- "See" that 4+4=8

  & reason that 4+4 equals 8
- Construe resultative counting as a curtaiment of counting individual objects
- Construe 'counting on' and 'counting back' as extensions of resulative counting

Van Hiele example: the concept 'rhombus' in geometry

- Students do not see a square as a rhombus



square rhombus

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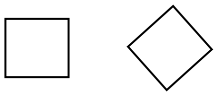
Van Hiele example: the concept 'rhombus' in geometry



Rhombus as a mathematical object:
junction in a network of mathematical relations:

- Sides are two by two parallel
- All sides have equal lengths
- Diagonals intersect orthogonal
- Facing angles are equal

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Consequences of the common view

- We perceive math an independent body of knowledge.
  - objects such as 'linear functions' as can be pointed to and spoken about
  - being able to talk and reason about these 'objects' unproblematically while interacting with others
- The body of knowledge only exist in the minds of teachers and textbook authors

➢ How can students connect to a body of knowledge that does not exist for them?

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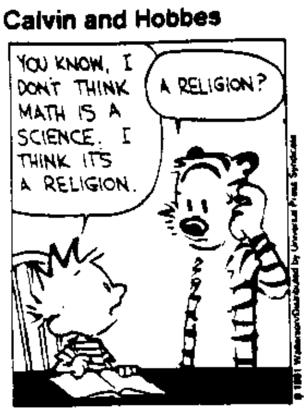

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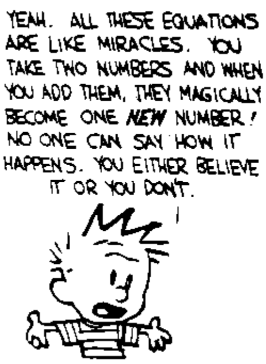

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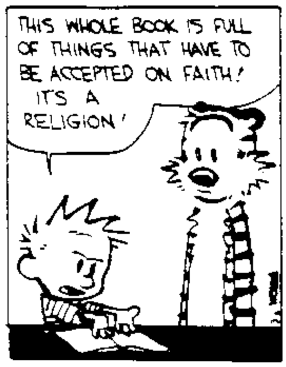

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Consequences of the common view

- Some people manage to reinvent mathematics even if it is not taught that way (They may advocate "Learn first, understand later")
- Most don't, they learn definitions and algorithms by heart →
  - Problems with applications
  - Problems with understanding
  - Math anxiety

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Need for a bottom-up process of concept formation

- The mathematics of the mathematicians cannot be conveyed by explanations or definitions
- Students have to go through a process of concept formation, within which operational conceptions precede the structural conceptions
- expanding common sense (Freudenthal)

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Students have to construct or reinvent mathematics

People construct their own knowledge

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Students have to construct or reinvent mathematics

- Constructivism: 'Students construct their own knowledge' →

  (…) students construct their ways of knowing in even the most authoritarian of instructional situations. (Cobb, 1994, p. 4)

➢ The question is not, Should the students construct?, but, What it is that we want the students to construct?

  or: "What do we want mathematics to be?" →

  Mathematics as a human activity (Freudenthal)

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Hans Freudenthal

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Students have to construct or reinvent mathematics

- Freudenthal: mathematics as a human activity:
  - Solving problems,
  - Looking for problems
  - Organizing subject matter

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Freudenthal (1971, 1973)
‘Mathematics as a human activity’

Mathematics as the activity of mathematicians that involves solving problems, looking for problems, and organizing subject matter;

- mathematical matter, or
- subject matter from reality

(mathematizing: organizing subject matter from a mathematical point of view)

Final stage: axiomatizing (“anti-didactical inversion”)

To engage students in mathematics as an activity

Students have to be supported in inventing mathematics → guided reinvention

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RME THEORY

- Generalizing over local instruction theories → domain-specific instruction theory for realistic mathematics education, RME, Treffers (1987)
- Later, RME theory was recast in terms of design heuristics:
  - guided reinvention,
  - didactical phenomenology,
  - emergent modeling

(Gravemeijer, 1989)

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Guided Reinvention

- Identify starting points that are experientially real for the students
- Identify the end points of the reinvention route

Design heuristics:

1. Look at the history of mathematics (potential conceptual barriers, dead ends, and breakthroughs)
2. Look for informal solutions that ‘anticipate’ more formal mathematical practices

→ potential reinvention route

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Addition and subtraction up to 20

<u>Guided reinvention</u>

- Starting points: informal solution procedures in contexts
- doubles, five- and ten-referenced number relations (fingers)
- End points: flexibel use of number relations; derived facts

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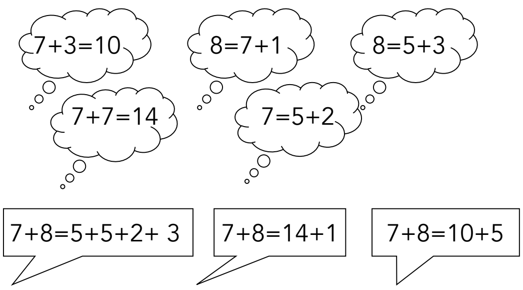
Derived facts e.g 7+8=...



Didactical Phenomenology

- Phenomenology of mathematics:
  Analyze how mathematical ‘thought-things’ (concepts, procedures, or tools) organize certain phenomena.
- Envision how a task setting may create the need to develop the intended thought thing
  → starting point for a reinvention process

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Addition and subtraction up to 20

Didactical phenomenology

Combine, change, compare

HF, not just structuring and combining sets of objects (cardinal)

- Also counting events, measuring, …(ordinal)
- Coordinating cardinality and ordinality:
- Problems that are stated cardinally (4 marbles and 3 marbles) are solved ordinally (counting on)

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Emergent Modeling: Circumventing the learning paradox

- External representations do not come with intrinsic meaning
- "Learning paradox" (Bereiter, 1985):

  How is it possible to learn the symbolizations, which you need to come to grips with new mathematics, if you have to have mastered this new mathematics to be able to understand those symbolizations?

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Circumventing the learning paradox

Emergent modeling:

- A dynamic process in which symbolizations and meaning co-evolve ⇔ history (Meira)
- Modeling as a student activity in service of the learning process

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Emergent modeling

- Starting point: modeling (situated) strategies
  - ➢ The model derives its meaning from the context it refers to
- Shift in attention; developing mathematical relations
  - ➢ The model starts to derive its meaning from a framework of mathematical relations; becomes a model for mathematical reasoning

"A model of informal mathematical activity becomes a model for more formal mathematical reasoning"

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" Model"

- Actually a series of sub-models
- From the perspective of the researcher/ designer, the series of sub-models constitute an overarching model.
- This overarching model co-evolves with some new mathematical reality

➢ Imagery: activities with new sub-models signify earlier activities with earlier sub-models for the students

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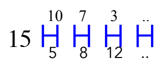
Addition & subtraction up to 20

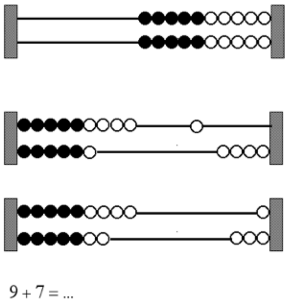
Emergent Modeling

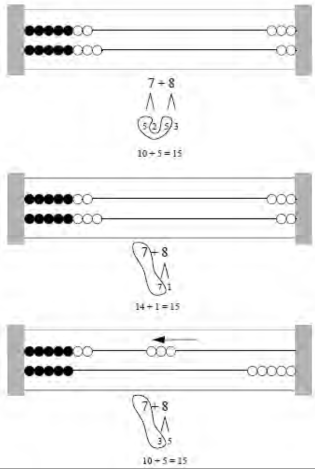
- Doubles
- Five & ten referenced
- Contexts, coins, ruler, .. double decker bus



- Arithmetic rack as a means of support; scaffolding &reasonig

Addition & subtraction up to 20





Nashville, with Cobb, Yackel, Whitenack

Arithmetic rack as a means of scaffolding & communicating

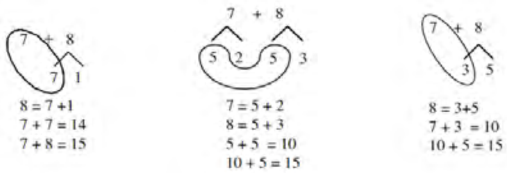
- Students realize that 7=5+2 and 8=5+3, and visualize that on the arithmetic rack

- They realize that 5+5=10, or 7+7=14, or 8+8=16

Model

- Arithmetic rack, *model of* ways passengers are seated in the double decker bus

- Shift to *model for* reasoning about number relations



Mathematics innovation in the Netherlands

- Curriculum innovation in the Netherlands; Little room for teacher professionalization
- Curriculum innovation via textbooks
- RME innovation in the Netherlands not enacted as intended (Gravemeijer et al, 1991)

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Research on mathematics innovation in the Netherlands

- National surveys PPON
  - decline in operations; standard algorithms
  - improvement in number sense, global arithmetic
- Math Wars
- Research on
  - Subtraction up to 100 (Kraemer, 2011)
  - Multiplication of fractions (Bruin-Muurling,2010)
  - Algebra (Stiphout, 2011)

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Subtraction upto 100
J-M. Kraemer (2011)

– computing methods:
- jumping; or skip counting, e.g.,
  65 – 38 = …, via 65 – 30 = 35, 35 – 5 = 30, 30 – 3 = 27
- splitting; splitting tens and ones, e.g.,
  68 – 45 = …; 60 – 40 = 20, 8 – 5 = 3, answer 20 + 3 = 23
- reasoning; deriving number facts using arithmetic properties, e.g.,
  62 – 48 equals 64 – 50 = 14; or
  62 - 48 = … via 62 – 50 = 12, thus 62 - 48 = 12 + 2 = 14)
- knowing; reproducing known facts.

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4th. PPON 2001 → 3rd. PPON 2005, no improvement

- Jumping ⬄ Treffers, (1991)
  – most used method
  – most effective method
  – applied flexibly.
- Splitting
  – many incorrect answers
- Reasoning
  – many incorrect answers

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problems

- buggy algorithm (e.g. 62 – 48 = 20 + 6)
- incorrect forms of splitting
- incorrect forms of reasoning
- tasks that involved bridging ten difficult

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- a limited understanding of the inverse relationship and a lack of proficiency in coordinating tens and ones.
- In other words, the students acquired restricted set of computing methods and reached only a limited level of conceptual understanding.

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Textbooks +,- ≤20

- a sound grounding of particular informal strategies, however,
- a continuation towards a higher level of understanding is missing
- attention for splitting methods and reasoning methods is rather sparse

⬄ the appeal of methods that routinely produce correct answers.

- students may have started to experiment with splitting and reasoning methods, while lacking a solid conceptual basis

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Multiplying Fractions
(Geeke Bruin-Muurling, 2010)

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9th graders ($N = 347$) (higher general secondary education, and pre-university education)

Topic	Successful
adding or subtracting fractions with unrelated denominators	only the best students
multiplication, only tasks such as 4/7 part of 35 euro, and 5 × 4 1/5	half of the students
division problems in context	half of the students
bare division items	one quarter of the students

The students did not grasp the underlying big ideas, such as unit, fraction as a number, and the relation between fractions, multiplication, and division.
The majority did not grasp the *relation between fractions and the operations of multiplication and division*.

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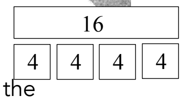
Multiplying Fractions, textbooks
(Geeke Bruin-Muurling, 2010)

- Primary school textbooks aim at number-specific solution methods (end of 6th. Grade)
- number-specific, context related, measures ("labeled numbers")

- 16 x ¾ ⇔16 cartons of cream of ¾ liter
  - ➢ repeated addition ¾ + ¾ + …
- ¾ x 16 ⇔ "¾ part of 16" →
  - ➢ dividing by 4 first 16:4=4; 3x4=12



Meaningful procedures grounded in the experiential reality of the students

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Number-specific procedures tied to contexts

- However, the next step is not taken: Why is it that 16 x ¾ equals ¾ of 16?
- Or that ¾ + ¾ + … ¾ equals [16:4=4] x 3? Students reason at the level of labeled quantities, "liters", "Gallons", …
- To understand the rule they have to reason at the level of numbers as objects

→ problems in transition primary-secondary

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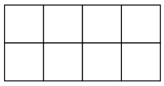
Secondary Education, new procedure with bare numbers

- A general rule for bare numbers (mathematical objects)

fraction x fraction = nominator x nominator / denominator x denominator



¾ x ½

picture as "proof"

- Additional procedures → confusion

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What is lacking

- From number-specific procedures to general rules
- Generalizing & formalizing

e.g. reasoning about why,
adding ¾ sixteen times (⇔ 16 x ¾),
and
taking 3 times ¼ of 16 (⇔ ¾ x 16)
gets you the same answer.

This kind of reasoning is lacking in Dutch primary and secondary schools

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- In other words, the students acquired restricted set of computing methods and reached only a limited level of conceptual understanding.

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Algebraic Skills

- Research in the Netherlands (van Stiphout, 2011)
- Conceptual tasks difficult for secondary school students
- Tasks such as:

  If $a\sqrt{b} = 1 + 2a\sqrt{1 + b}$, then a = ...

  Global Substitution Principle (Wenger)

  $a\sqrt{b} = 1 + 2a\sqrt{(1 + b)} \Leftrightarrow 2v\ P = 1 + v\ Q$

  Grade 10, 0% correct

  Grade 11, 1% correct

  Grade 12, 1% correct

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Algebraic Skills

- Solve: $(x - 5)(x + 2)(x - 3) = 0$

	2008	2009	
Grade 9,	4%	51%	correct
Grade 10,	29%	40%	correct
Grade 11,	37%	52%	correct
Grade 12,	47%	75%	correct

Students are familiar with 2 factors, e.g. $(x - 5)(x + 2)=0$

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Algebraic Skills

- Overall picture, the students acquired restricted set of computing methods and reached only a limited level of conceptual understanding.

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Algebraic Skills

- Textbooks: dual track: start conceptual, shift to procedural

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Research on mathematics innovation in the Netherlands

- Three studies showed:

  instructional sequences break off too early:
  - Students ground their mathematical understanding in situations that are experientially real to them → well-understood, situation-specific, solution procedures
  - Situation-specific solution procedures produce correct answers → teachers and textbooks capitalize on those procedures, routinizing

(Bruin-Muurling (2010), Kraemer (2011), & van Stiphout (2011))

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Research on mathematics innovation in the Netherlands

- The next step is lacking!
  - Reflecting on those procedures
  - Developing more sophisticated and generalized understandings
- 'Task Propensity'
  - The inclination of both teachers and textbook authors "to think of instruction in terms of individual tasks that have to be mastered by the students" (Gravemeijer et al. (2016)

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Task Propensity

- Task propensity counteracts inquiry math
  - focus on procedures that can quickly generate correct answers, instead of supporting students in coming to understand the underlying mathematics.
- Task propensity ⇔ views on learning

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Task Propensity

- Resnick and Hall (1998): the popular view ⇔ classic associative theories of learning:
  - Knowledge consists of connections between mental entities, and learning is a matter of creating and strengthening these bonds →
  - Frequent testing of individual items to see if the bonds are formed
- The idea of proficiency in terms of mastery of individual test items also permeates goal descriptions in curriculum documents

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More advanced conceptual mathematical goals

- Goal description in line with researchers' views on learning mathematics
  - constructing mathematics ⇔ transitions: processes → mathematical objects,
    - which in turn become subject to new processes (Sfard, 1991, Tall, 2008, Freudenthal, Dubinsky,…)
- Goals of mathematics education in terms of mathematical objects
  - which derive their meaning from frameworks of mathematical relations

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Advanced goals Addition & subtraction 100

- Numbers as objects in networks of number relations
- Unitizing; units of 10 → e.g. 63 = 6 x 10 + 3;

  63 = 60 + 3; 63 = 50 + 13 and 63 = 70 – 7 etc.

  respectively 60 + 3 = 63 etc.
- Expanding to making structures such as

  63 = 40 + 23 etc.

  63 = 37 + 26

  63 = 32 + 31 etc.
-

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Advanced goals Addition & subtraction 100

- Interconnectedness addition & subtraction

  63 — 23

  63 — 40

  Instead of procedures, working with number relations
- NEXT Level: Sums and differences as objects
- E.g comparing "65+17" and "65+7"

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Advanced goals Multiplication of Fractions

1. From fractions, with identifiable units (pizza's, meters, …)
2. To rational numbers, as mathematical objects

   ⇔ number relations, e.g.

   ¾ = ¼ + ¼ + ¼

   = 3 x ¼

   = 1 – ¼

   = ½ + ¼; ..)

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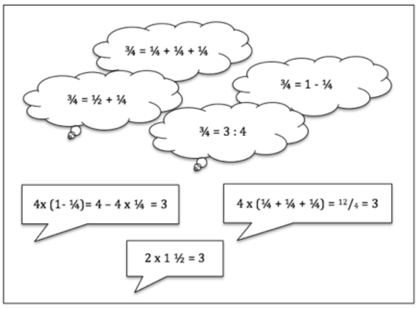
3 x ¾



Framework of number relations; derived facts

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Advanced goals
Multiplication of Fractions

Relations between operations; multiplication, division, fractions, proportion, e.g.:

¾ ⇔ 4 ÷ 3
16 x ¾ = 16 x (3 ÷ 4)
¾ x 16 = (3 ÷ 4) x 16

Structural ⇔ operational (Sfard, 1991)
¾ as an object, rational number
¾ as a computational prescription 4 ÷ 3

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Advanced goals
Multiplication of Fractions

- Eventually, students have to come to see a product, such as,16 x ¾ and ¾ x 16 as one thing (a mathematical object in and of itself)
- Splitting & (re)combine objects
- E.g. see a^2/b^2 as both axa/bxb and as a/b x a/b.
  For instance in $a^2/b^2 - 1 = (a/b)^2 - 1 = (a/b - 1)(a/b + 1)$

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Advanced goals, Algebra

Structure sense, symbol sense (Arcavi, --)
(test items) ⇔ test items
Structural ⇔ procedural (Sfard, 1991)
Object ⇔ process

- 2x+7 -> computational prescription
  -> object
  $(2x+7)^3$
  2x+17 is 10 more than 2x+7
  4x+14 is twice 2x+7

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Shortcommings

In summary

- The students acquired restricted set of computing methods and reached only a limited level of conceptual understanding.
- 'Task Propensity': focus on individual tasks
- More advanced conceptual mathematical goals are not addressed
  - Not in textbooks
  - Nor in tests or curriculum documents
- Inquiry classroom culture not sufficiently developed (earlier research)

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NL not unique

- The math ed community has been trying to get problem-centered, inquiry math implemented for several decades now.
- We have learned about the obstacles, but also about determining factors.

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critical points concerning the enactment of inquiry mathematics:

- Establishing adequate social norms and socio-mathematical norms (Cobb & Yackel; 1996)
- Fostering task orientation over ego orientation
- Cultivating mathematical interest
- Framing topics for discussion
- Designing and adapting hypothetical learning trajectories

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Social norms & socio-mathematical norms

Traditional school-math social norms:

- students have to come to grips with knowledge the teacher already has
- the teacher's role is to explain and clarify;
- the students' role is to try to figure out what the teacher has in mind.

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Social norms & socio-mathematical norms

Inquiry classroom social norms:

- obligation to explain and justify one's solutions
- to try and understand other students' reasoning
- to ask questions if one does not understand
- challenge arguments one does not agree with.

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Social norms & socio-mathematical norms

Socio-mathematical norms relate to what mathematics is, .g.:

- what counts as a mathematical problem,
- what counts as a mathematical solution,
- what counts as a more sophisticated solution.

⇔ intellectual autonomy students

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Social norms & socio-mathematical norms

- To establish new social norms the teacher has to convince the students that, what is valued and what is rewarded, has changed.
- Use concrete instances of infringements on the new norms or exemplary behavior as opportunities to clarify the norms.

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Cultivating an inquiry classroom culture

- Asking for explanations
  - Please explain your answer
- Asking for clarifying questions
  - Who has a question for Jim?
- Pass the problem along
  - Who can answer Paula's question?
- Asking for a personal judgment
  - Ann says that it will cost $16.25, do you agree?
- Promoting that students listen and try to understand
  - Did you follow what he said, could you explain it to me?

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Cultivating an inquiry classroom culture

- How long too cook a turkey?
- The turkey is 24 lbs.
- Take 15 minutes per lbs.

  - Math in the City (Dolk & Fosnot)

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Cultivating an inquiry classroom culture

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Pro-active role of the teacher

- “Try to explain in such a manner that everybody can understand.”
- “Listen carefully, and see if you understand.”
- “Who thinks he can explain what Amber and Vicky tried to do?”
- “Do you have something to add?”
- “Without telling them how many hours could you explain to them how they could figure that out?”
- “Great question, did you understand …”
- “Tell them ..”
- “Did you hear what he said ?”

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Task/Ego orientation
(Jagacinski & Nicholls, 1984)

Students also have to be willing to invest effort in solving mathematical problems, discussing solutions, and discussing the underlying ideas

- Ego orientation; student is very conscious of the way he or she might be perceived by others
- Task orientation; student’s concern is with the task itself
- Cultivating task orientation:

  creating a classroom culture, where students measure success by comparing their results with their own results earlier.

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Cultivating mathematical interest

mathematical interest ≠ pragmatic interest

- asking questions such as:
  - What is the general principle here?
  - Why does this work? Does it always work?
  - Can we describe it in a more precise manner?
- showing a genuine interest in the students’ mathematical reasoning
- mathematical interest prerequisite for reinvention

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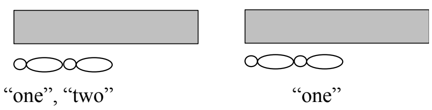
Framing topics for discussion
(Cobb, 1997)

- identify the differences in mathematical understanding
- topics for whole-class discussions
- Two ways of measuring



- counting vs accumulation of distance

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Hypothetical learning trajectory (Simon, 1995)

- Constructivism: Teachers can influence their students' knowledge construction only in an indirect manner → anticipate, enact, observe, revise
- HLT: *Think through the mental activities the students might engage in as they participate in the envisioned instructional activities ⇔ learning goals*
- *investigate whether the thinking of the students actually evolved as conjectured; revise or adjust*

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Hypothetical learning trajectory (Simon, 1995)

- From "Teachers transmitting knowledge"
- To "Students inventing" → How to make students invent what you want them to invent?
- Indirect (HLT, Simon, 1995):
  - Anticipate on what students might be thinking when participating in an instructional activity (Simon)
  - ⇔ shifting from an observers´ point of view to an actors´ point of view

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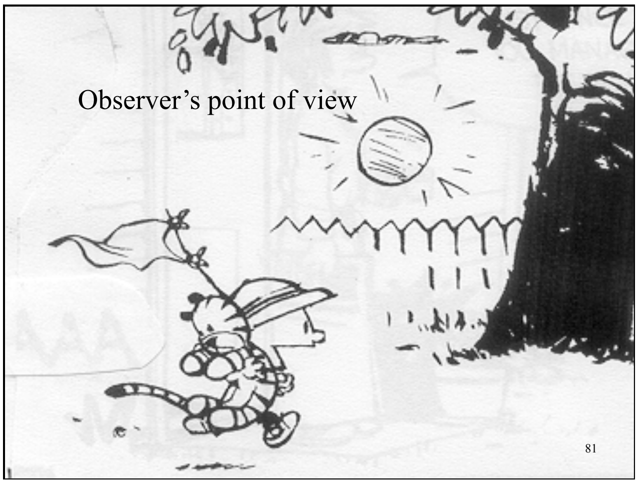

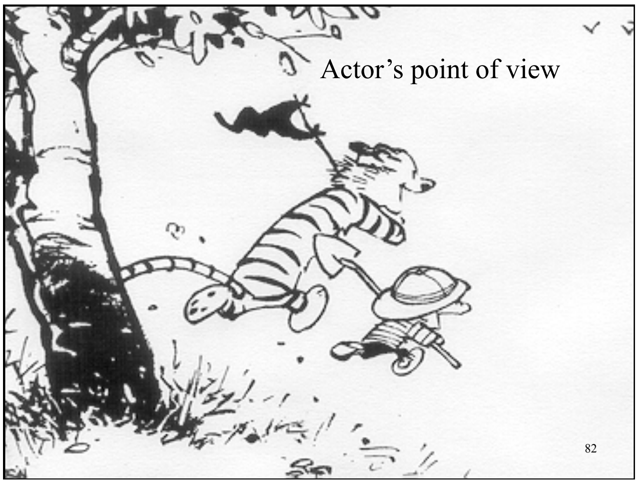

Hypothetical: try and revise

thought experiment

instructional activity

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What type of support should be offered to teachers?

- We should aim at offering teachers means of support for construing and revising HLT's
- By developing prototypical instructional sequences and for various topics (fractions, long division, …) & the corresponding Local instruction theories

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Local instruction theory
(Gravemeijer, 2004)

- Local instruction theory:
  - a theory about a possible learning process for a given topic,
  - with theories about the means of supporting that process (tasks, tools, classroom culture)
- Teacher support: LiT as framework of reference for designing HLT's
  - for this teacher, these students and at this moment in time.

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Teacher Professionalization

- Fullan (2006), most innovations do not have a lasting impact
- Teachers do not get feedback on what they do in their own classrooms
  - They might be doing marvelous things (or the opposite) without knowing it
- Needed: Collaboration between teachers that includes visiting each other's classroom.
- Groups of teachers who commit themselves to improving their teaching working collaboratively

  ⇔ Japanese lesson studies (Stigler & Hiebert, 2009)

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Teacher Professionalization

- Literature on teacher change: Key, teacher ownership & intrinsic motivation (…)
- Lesson Studies:
  - Teachers may work together to elaborate on a given local instruction theory in order to make it work in their own classrooms
  - Instructional designers will also have to invest in designing means of support for lesson-study-type of activities

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Strategy

- Put a spot on the horizon (ambition)

  ⇔ support of the wider society

  ⇔ digital society
- Zone of proximal development teachers
- E.g. starting with supplementary activities

  Cathy Fosnot, "Contexts for learning"
  - Teacher development, workshops, coaching
  - Instructional materials to experiment with (one topic, 2 weeks)

Conclusion

<u>RME: mathematics as an activity</u>

- Organizing/mathematizing
- Starting points exprientially real
- Learning as constructing ⇔ reinvention
- Mathematics – growing common sense
  - Guided reinvention
  - Didactical phenomenology
  - Emergent modeling

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Conclusion

<u>Curriculum innovation NL</u>

- Texbooks: instructional sequences brake of too early ⇔ Task propensity
  - Textbooks
  - Teachers
  - Curriculum documents
  - Tests
- ➢ More advanced conceptual goals
  - Mathematical objects; networks of mathematical relations

Conclusion

- Need for teacher learning
  - social norms
  - task orientation
  - mathematical interest
  - topics for discussion
  - hypothetical learning trajectories
- Need for new means of support
  - Local instruction theories instead of scripted tekst books
  - Lesson studies; supplementary activities
- Need for support of the wider society

Thank You

Subtraction up to 100

300 Grade 3 students

- jumping use 57% - 74%
  - 82% - 91% correct
- splitting
  - 65% - 42% correct
- reasoning
  - 50% - 31% correct